Chernoff bound

Theorem: Let $X_1, \dots, X_n$ be independent random variables that lie in interval $[0,1]$ with probability one. Let $p = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i]$.
For any $\varepsilon > 0$,

$$\Pr\left[\frac{1}{n}\sum_{i=1}^{n} X_i \geq (1+\varepsilon)p\right] \leq \left[\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}}\right]^{np}$$

Proof: Let $p_i = \mathbb{E}[X_i]$
Chernoff bounding technique.

$$\Pr\left[\frac{1}{n}\sum_{i=1}^{n} X_i \geq (1+\varepsilon)p\right]$$

$$= \Pr\left[e^{s\sum_{i=1}^{n} X_i} \geq e^{(1+\varepsilon)snp}\right] \qquad \boxed{s>0}$$

$$\leq \frac{\mathbb{E}\left[e^{s\sum_{i=1}^{n} X_i}\right]}{e^{(1+\varepsilon)s\mu p}}$$

$$= \frac{\prod_{i=1}^{n} \mathbb{E}\left[e^{sX_i}\right]}{e^{(1+\varepsilon)s\mu p}} \qquad \text{(by independence)}$$

$$\leq \frac{\prod_{i=1}^{n} 1-p_i + p_i e^{s}}{e^{(1+\varepsilon)s\mu p}} \qquad \text{(HW \#1)}$$

$$= \frac{\prod_{i=1}^{n} 1+p_i(e^{s}-1)}{e^{(1+\varepsilon)s\mu p}}$$

$$\leq \frac{\prod_{i=1}^{n} e^{p_i(e^s-1)}}{e^{(1+\varepsilon)snp}}$$

$$\boxed{\boxed{1+z \leq e^{z}}}$$

$$= \frac{e^{(e^s-1)\sum_{i=1}^{n} p_i}}{e^{(1+\varepsilon)snp}}$$

$$= \frac{e^{(e^s-1)np}}{e^{(1+\varepsilon)snp}}$$

$$= \left(e^{(e^s-1)-(1+\varepsilon)s}\right)^{np}$$

Find s that minimizes the

last expression.

$$f(s) = e^s - 1 - (1+\varepsilon)s$$

$$f'(s) = e^s - (1+\varepsilon)$$

Solve $f'(s) = 0$:

$$e^s - (1+\varepsilon) = 0$$

$$s = \ln(1+\varepsilon)$$

$$\Pr[\cdots] \le \left(e^{(e^s-1) - (1+\varepsilon)s}\right)^{np}$$

$$= \left(e^{(1+\varepsilon-1) - (1+\varepsilon)\ln(1+\varepsilon)}\right)^{np}$$

$$= \left(\frac{e^{\varepsilon}}{(1+\varepsilon)^{1+\varepsilon}} \right)^{np}$$

$\blacksquare$